A 4-Dimensionalist Mereotopology

John G. STELL
School of Computing, University of Leeds

Matthew WEST
Shell Information Technology International Limited, London

Abstract. We develop a formal ontology within the four dimensionalist (4D) paradigm by showing how the algebraic description of spatial mereotopology – a Boolean algebra equipped with a connection relation – can be enriched to provide a mereotopology in which the entities are spatio-temporal, rather than spatial, regions. With the 4D approach it is natural to identify a period of time with all of space during that time, and thus to model temporal relations by relations between spatio-temporal regions. Building on the concept of a Boolean connection algebra, we propose two additional structures: an historical closure operator, and a pair of further operators, called pre-history and post-history, which form a Galois connection. We provide axioms for these operators and show how their properties relate to Muller's axiomatization for qualitative spatio-temporal reasoning.

Keywords: Formal Relations; Knowledge Representation; Identity and Change; Ontology of Physical Reality

1 Introduction

1.1 Algebraic Structure in Mereotopology

One approach to mereotopology is to provide a theory having a mereological part – usually a Boolean algebra – and a topological part, which consists of a binary relation of connection. The partial order in the Boolean algebra, denoted $\leqslant$ below, models the idea that one region may be part of another. The connection relation formalizes the notion that one region is connected to another. In this approach parthood is taken as a primitive, rather than defined from connection, but the usual relationship between parthood and connection appears as a theorem instead of being a definition. There are various possibilities for the axioms satisfied by connection, such as those of Stell [1] or those of Vakarelov et al [2]. This approach to mereotopology can fairly be called 'algebraic'. It takes advantage of well-known algebraic structures, such as a Boolean algebra, or the relation algebras used by Düntsch [3, 4], rather than axiomatizing the entire theory *ab initio* using first order logic. This latter, 'logical', approach is seen in the original Region-Connection Calculus (RCC) papers [5, 6] and in Muller's spatio-temporal mereotopology [7].

One significant advantage of the algebraic over the logical approach is that building the theory in a modular way, using well known algebraic structures as the building blocks, allows the vast amount of knowledge about these structures to be used to contribute to the development of mereotopology. As noted above there are several algebraic approaches to a mereotopology of spatial regions. However, there does not seem to be an algebraic account of the mereotopology of spatio-temporal regions. Models of a system such as RCC can indeed be three dimensional, and we might like to think of this as being two spatial dimensions and one temporal. However, this does not provide a spatio-temporal meretopology since the structure provides no means of referring to specifically spatial or temporal relations between the entities. To pick out these additional properties requires additional structure, and additional axioms. In this paper we show how the algebraic approach to meretopology can be extended to provide the structure needed to make purely spatial and purely temporal distinctions. In particular, we identify two additional structures – a topological closure operator, and a pair of operators forming a Galois connection – which provide the apparatus necessary to define temporal relations between spatio-temporal entities.

1.2 Structure of the paper

In section 2, we review the ontological background to the work explaining how the 4D paradigm underlies use of mereotopology in the paper. That section also situates the work in the wider context including ISO 15926 and the IEEE Standard Upper Ontology Working Group. The notion of a Boolean Connection Algebra is recalled in section 3, and we consider some of the notions of part which our formal theory needs to be able to handle. The technical core of the paper is found in sections 4 and 5. The first of these introduces our historical closure operator and shows how it allows the definition of historical connection, historical part and temporal part. Theorems establishing properties of these are given, but there is not sufficient space to give full proofs of all of the results. Section 5 introduces the second of the two building blocks of our algebraic approach: a Galois connection. From the Galois connection we define a temporal precedence relation, and its key properties are established. Our approach has been influenced by Muller's paper [7], which has a logical rather than algebraic basis, although it is not our intention to provide an equivalent system. A comparison of our axiomatization with Muller's is given in section 6. We end in section 7 with conclusions and suggestions for further work.

2 Ontological Background

2.1 3D and 4D approaches to ontology

In principle, there are infinitely many ways in which we can model the world, so it is perhaps surprising that there are two main approaches, with on the whole minor variations, that dominate the literature. We will call these the 3D paradigm and the 4D paradigm, though they are also known as endurantism, and perdurantism. A 4D ontology treats all individuals - things that exist in space-time - as spatio-temporal extents, i.e. as 4D objects.

The principles of the 4D paradigm are:

1. Individuals exist in a manifold of 4 dimensions, three space and one time. So things in the past and future exist as well as things in the present.

2. The four dimensional extent is viewed from outside time rather than in the present.

3. Individuals extend in time as well as space and have both temporal parts and spatial parts.

4. When two individuals have the same spatio-temporal extent they are the same thing. However this principle is not always insisted on.

5. The object over its whole life is the object of primary interest.

Thus a 4D object is not (usually) wholly present at a point in time, but its whole is extended in space as well as time. The object at a point in time is a temporal part of the whole. Change is naturally expressed through a 4 dimensional classical mereology, which Simons [8], in his seminal work, describes in one page. A good description of, and argument for, the 4D paradigm can be found in Sider [9].

A 3D ontology treats physical objects (roughly things you can kick) as 3D objects (sometimes called continuants) that pass through time. The principles of the 3D paradigm are:

1. Physical objects are 3-dimensional objects that pass through time and are wholly present at each point in time.
2. Physical objects are viewed from the present. The default is that statements are true now.
3. Physical objects do not have temporal parts.
4. Different physical objects may coincide.
5. The object-at-a-point-in-time is the object of primary interest.

To talk about an object at different times it is necessary to time index statements in some way, e.g. X at t. A 3D ontology also has 4D objects in it. These cover activities, such as:

- a football match - which clearly has temporal parts such as the first half and the second half,
- a living process - a persons life, rather than the physical person passing through time.

The 3D approach corresponds well with the way that language works. Language has a focus around here, now, you and me as a context, and on the current state of affairs. This leads to efficient communication under the most common circumstances. On the other hand dealing with change is relatively problematic. Simons [8] requires several chapters to explain how objects change over time in a 3D ontology.

2.2 Which paradigm?

It should be noted that there is much heat but no consensus on whether one or other of these approaches is right or wrong, better or worse. What is clear is that the 3D and 4D paradigms cannot be merged into a single canonical approach, since they are contradictory, with one requiring physical objects to have temporal parts, and the other forbidding them. On the other hand, it appears that what can usefully be said using one paradigm can generally be said using the other . We therefore believe that a judgement between the two - if such a judgement is even appropriate - will eventually be one of better/worse rather than right/wrong. An informed judgement will require that each paradigm is worked out into a full set of axioms. Only then will it be possible to make considered judgements of elegance and efficiency of the different paradigms.

2.3 Implications for Information Systems

In the early '90s the Process Industries identified a requirement to exchange the design information for Process Plants in electronic rather than paper form, and for structured information to be exchanged as data. A number of consortia were established to support the achievement of this aim, and these came together in an organisation called EPISTLE to develop standards to support this requirement. The standards are:

- ISO TS 18876:2003 - Integration of industrial data for exchange access and sharing (IIDEAS). This consists of two parts: Part 1 provides an architecture for data integration; Part 2 defines a methodology.
- ISO 15926 - Integration of life-cycle data for process plants including oil and gas facilities. The standard defines a data model, reference data, and exchange templates to support the integration and exchange of engineering design data, although the standard has somewhat wider applicability. The data model has been published as an International Standard as ISO 15926-2:2003.

Most data models are developed without consideration of ontological principles, however, this one was defined in terms of the 4D paradigm as outlined above. The 4D paradigm was eventually chosen because of its rigour. The data model is used by a wide range of people, and it was discovered that concepts were being interpreted in a variety of unintended ways by different users and developers. A common interpretation is important between parties who are intending to exchange information. Adopting the 4D paradigm was found to significantly reduce both the possibility of misinterpretation, and of incompatible extensions being developed.

However, a data model, whilst it might define concepts that support ontological principles, does not generally capture more than a portion of them formally. So the next stage is to take this work and add formal axioms missing from the data model to enrich the model towards something that would support some reasoning. The forum in which this work is progressing is the IEEE Standard Upper Ontology Working Group, and this paper is a contribution to this work.

3 Spatio-Temporal and Spatial Mereotopology

3.1 Mereology: Parts, Temporal Parts, and Historical Parts

In a mereotopology of space, the basic entities are spatial regions. We may conceive of these as being 3-dimensional, but in many applications we deal with 2-dimensional regions. The formal account consists of a theory of parts of these entities together with a connection relation between them. Once we view entities as not just spatial but having one of their dimensions temporal, various kinds of part can be distinguished. Although we work within the 4D paradigm, our mereotopology says nothing about the dimension of the entities – they could have one, two or more spatial dimensions, but they do have a distinguished dimension which we think of as time. Once we have entities which are both spatial and temporal, there are several separate notions of part that can be distinguished, and thus the mereological component of the theory is not obvious. Three that concern us are as follows.

spatio-temporal part If our spatio-temporal regions are 4D, this is just the usual notion of 4D part. For example, a tree is a spatio-temporal extent from germination until death. The bark of the tree is a spatio-temporal part of the tree.

temporal part The bark is not a temporal part of the tree. There are other parts of the tree which exist at the same time as the bark but which are not parts of the bark. A temporal part is a spatio-temporal part which consists of the whole over some period(s) of time. For example, the tree in winter is a temporal part of the tree. For more on temporal parts, see Meixner [10, p189,203ff] and Gallois [11, p255ff].

historical part An historical part of the tree is a spatio-temporal entity, but need not be a spatio-temporal part. Something is an historical part when its lifetime is included in the whole. An alternative terminology would be temporal subsumption rather than historical parthood. For example a given leaf on a different tree can be an historical part of our particular tree.

In spatial mereotopology, the mereological basis is that of a Boolean algebra. That is, we assume we have operations of union of parts (denoted $+$) and intersection of parts (denoted $\cdot$). The union of two parts is again a part, and similarly for the intersection. There is also an operation of complement, the complement of x being denoted x^*. We can also refer to the universe, denoted u, and the empty spatio-temporal region, denoted null. It should be noted that having an empty region is controversial, but it is used here in the formal theory as it greatly simplifies the formal work. It does not imply an ontological commitment to the existence in a real sense of an empty region.

3.2 Connection: Spatial and Spatio-Temporal

As noted in the introduction, various systems of spatial mereotopology have been proposed. The basis we choose is that of a Boolean connection algebra defined by Stell [1]. The advantage of this approach is the separation of the mereological basis from the topological superstructure. Intutitively, connection corresponds to overlapping or touching. First we recall the definition.

Definition 1. *Let* $A = \langle A; \mathsf{null}, \mathsf{u}, ^*, +, \cdot \rangle$, *be a Boolean algebra with* $|A| > 2$, *and let* $R = A - \{\mathsf{null}\}$, *and* $R_- = A - \{\mathsf{null}, \mathsf{u}\}$. *If* A *is equipped with a binary relation* C *of connection, satisfying the following axioms, then the pair* $\langle A, \mathsf{C} \rangle$ *is a* ***Boolean Connection Algebra****.*

A1. $\forall x \in R \cdot x \,\mathsf{C}\, x$.
A2. $\forall x \in A \cdot x \,\mathsf{C}\, y \implies y \,\mathsf{C}\, x$.
A3. $\forall x \in R_- \cdot x \,\mathsf{C}\, x^*$.
A4. $\forall x \in A \cdot x \,\mathsf{C}\, (y + z)$ *if and only if* $x \,\mathsf{C}\, y$ *or* $x \,\mathsf{C}\, z$.
A5. $\forall x \in R_- \cdot \exists y \in R \cdot x \not\mathsf{C}\, y$.

This differs from the formulation in [1] only in that axiom A4 quantifies over all elements of the algebra A including null. The significance of this modification is that it then follows from the axioms that the null region is not connected to anything, and that no region other than null has this property. This fact is needed in the proof of Theorem 1 below. The partial order in the Boolean Algebra, denoted $\leqslant$, is definable by $x \leqslant y$ iff $x + y = y$, and this models the notion of part between regions. This parthood relation is often denoted P in the literature on the Region-Connection Calculus, although in RCC the relation P is defined from C whereas here parthood is primitive. The property $x \leqslant y$ iff for all $z, z\mathsf{C}x$ implies $z\mathsf{C}y$ can be proved from the above axioms.

As with parthood, discussed in section 3.1, the extension from spatial to spatio-temporal offers a number of options and it is possible to distinguish different kinds of connection. In section 4 we define a notion of historical connection from the historical closure operator, rather than introduce any new connection primitives.

3.3 Models

There is more than one way in which a set of axioms for qualitative reasoning may be used. One possibility is to have some particular structure or structures in mind for the interpretation of the axioms. For example we could take the Boolean algebra of regular closed regions in the plane as the set of all regions. Then the primitives provided by the language are used as a restricted way of making statements about the intended interpretations. An alternative is to study the axioms without taking a fixed interpretation for the models of the axioms. This latter is closer to, for example, the usual mathematical enterprise with, say, the theory of topological spaces or of groups, where although particular interpretations inform the choice of the axioms, there is no notion of intended interpretation – only of model of the axioms. In spatial reasoning the origins of connection-based mereotopology lie in the work of Whitehead [12] and the enterprise can be seen [12, p416ff] as attempting to characterise aspects of space itself, rather than to assume that space had some given mathematical structure and to use the notion of connection as a limited way of accessing it. In this paper we do not assume a particular intended interpretation, but the diagrams indicate one possible simple model with time and space each being one dimensional.

4 Historical Closure

4.1 Statement of Axioms

This section introduces our concept of historical closure. This powerful notion allows us to define temporal part, historical part, and historical connection among other constructions. To each spatio-temporal region x we associate its historical closure, $\langle x \rangle$. This historical closure is itself a spatio-temporal region, and consists of all space and time during the existence of x.

Space and time can each be visualised as one-dimensional, time being vertical and space being horizontal. It is important to note that this is only a technique for drawing useful diagrams and the axioms do not constrain space to be one-dimensional. This is illustrated on the left hand side in figure 1. In this figure, a spatio-temporal entity is shown as a solid black region consisting of two disconnected parts; the associated regions produced by the operators introduced later are shown by various shadings, or by combinations of shadings. The historical closure, which we consider first, consists in figure 1 of the part shaded horizontally together with the part in solid black. Other parts of this figure are best understood by referring forward to the discussion in section 5.1.

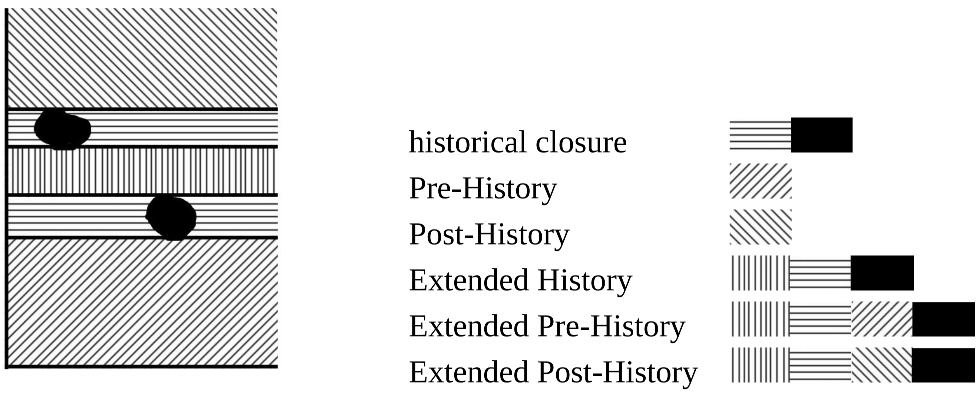

Figure 1: Visualization of Operators

The three axioms for historical closure are as follows.

LCL-1 $x\mathsf{C}y \implies x\mathsf{C}\langle y \rangle$

LCL-2 $\langle x \rangle\mathsf{C}\langle y \rangle \implies x\mathsf{C}\langle y \rangle$

LCL-3 $\langle\langle y\rangle^*\rangle = \langle y\rangle^*$

In both axiom LC-1 and axiom LC-2, the symmetry of C of course immediately implies that $\langle x\rangle \mathsf{C} y$ follows too. The third axiom expresses that the complement of an historically-closed region is itself historically-closed.

These three axioms are quite weak when taken on their own. For example, they do not prevent the interpretation that $\langle x\rangle = \mathsf{u}$ for $x \neq \mathsf{null}$, and $\langle \mathsf{null}\rangle = \mathsf{null}$. However, our system as a whole does not admit this trivial interpretation, as can easily be seen from lemmas 11 and 13 later in the paper.

4.2 Topological Closure Operators

We recall the definition of a topological closure operator on a Boolean algebras.

Definition 2. *A* **topological closure operator** *on a Boolean algebra, A, is a function $\Diamond : A \to A$ such that, for all $a, b \in A$,*

1.	$a \leqslant \Diamond a$,	*($\Diamond$ is increasing)*
2.	$\Diamond\Diamond a = \Diamond a$,	*($\Diamond$ is idempotent)*
3.	*i.* $\Diamond \mathsf{null} = \mathsf{null}$, *ii.* $\Diamond(a + b) = \Diamond a + \Diamond b$.	*($\Diamond$ is additive)*

Theorem 1. *The historical closure operator $\langle \_ \rangle$ is a topological closure operator on the Boolean algebra of spatio-temporal regions.*

Proof. It follows immediately from axiom LCL-1 and relationship between C and $\leqslant$ that $a \leqslant \langle a\rangle$. To show that $\langle a\rangle = \langle\langle a\rangle\rangle$ it is sufficient to show $\langle\langle a\rangle\rangle \leqslant \langle a\rangle$, but if $z \,\mathsf{C}\, \langle\langle a\rangle\rangle$ then $z \,\mathsf{C}\, \langle a\rangle$ by LCL-2 and the symmetry of C.

For the additivity, $z \,\mathsf{C}\, \langle a + b\rangle$ iff $\langle z\rangle \,\mathsf{C}\, (a + b)$ iff $\langle z\rangle \,\mathsf{C}\, a$ or $\langle z\rangle \,\mathsf{C}\, b$. But this is equivalent to $z \,\mathsf{C}\, \langle a\rangle$ or $z \,\mathsf{C}\, \langle b\rangle$, which happens iff $z \,\mathsf{C}\, \langle a\rangle + \langle b\rangle$. To show that $\langle \mathsf{null}\rangle = \mathsf{null}$, we have $z \,\mathsf{C}\, \langle \mathsf{null}\rangle$ iff $\langle z\rangle \,\mathsf{C}\, \mathsf{null}$, but this never happens as no region is connected to null, so no region can be connected to $\langle \mathsf{null}\rangle$, and hence this can only be null itself. □

A useful property, which holds for any topological closure operator, is monotonicity. That is, if x is a part of y then $\langle x\rangle$ is a part of $\langle y\rangle$. A consequence of this (since $x \cdot y$ is a part of x) is that $\langle x \cdot y\rangle \leqslant \langle x\rangle \cdot \langle y\rangle$.

4.3 Historical Connection

We define historical connection, by saying that two entities are historically connected if their historical closures are spatio-temporally connected. We use $\between$ to denote historical connection, following [7].

Definition 3 (Historical Connection). *$x \between y$ if and only if $\langle x\rangle \,\mathsf{C}\, \langle y\rangle$.*

Theorem 2. *Historical connection has the following properties*

1. *It is reflexive and symmetric*
2. *$x \,\mathsf{C}\, y$ implies $x \between y$*
3. *$(x + y) \between z$ if and only if $x \between z$ or $y \between z$*

Proof. Reflexivity and symmetry are immediate from the corresponding properties of the C relation. That spatio-temporal connection implies historical connection follows from LCL-1 and the symmetry of C. To prove the third part of the theorem, we have $\langle z\rangle \,\mathsf{C}\, \langle x + y\rangle$ iff $\langle z\rangle \,\mathsf{C}\, (\langle x\rangle + \langle y\rangle)$ from Theorem 1. This is equivalent to $\langle z\rangle \,\mathsf{C}\, \langle x\rangle$ or $\langle z\rangle \,\mathsf{C}\, \langle y\rangle$ as required. □

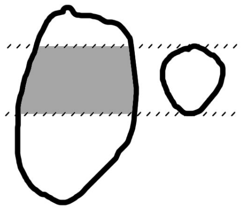

Figure 2: Temporal Part

4.4 Historical Part

We use the notion of historical connection to define historical part.

Definition 4 (Historical Part). $x \leqslant_{\mathsf{hist}} y$ *if and only if* $\langle x\rangle \leqslant \langle y\rangle$.

Some basic properties of this relation, which show its connection with spatio-temporal parthood, are in the next two lemmas.

Lemma 3. $x \leqslant_{\mathsf{hist}} y$ *if and only if* $\forall z\,(z \between x \implies z \between y)$ *if and only if* $\forall z\,(\langle z\rangle \mathsf{C} x \implies \langle z\rangle \mathsf{C} y)$.

Lemma 4. $x \leqslant y$ *implies* $x \leqslant_{\mathsf{hist}} y$.

4.5 Temporal Part

Definition 5. *For regions x and y, the temporal part of x induced by y is defined to be $x \cdot \langle y\rangle$.*

The intuition behind this is illustrated in figure 2 where x is the larger region on the left, y is the smaller region on the right, and $x \cdot \langle y\rangle$ is the shaded portion of x. A special case of the definition is when y is a temporal part of x, that is when the temporal part of x induced by y is y itself. Thus y is a temporal part of x exactly when $x \cdot \langle y\rangle = y$.

Theorem 5.

$$\langle x\rangle \cdot y = x \text{ if and only if } \begin{cases} x \leqslant y, \text{ and} \\ \forall z\; z \leqslant y \wedge z \leqslant_{\mathsf{hist}} x \implies z \leqslant x \end{cases}$$

Proof. Taking the left to right direction first. Since $x \leqslant \langle x\rangle$, we can use $\langle x\rangle \cdot y = x$ to get $x \cdot y = x$ so $x \leqslant y$. Now suppose that $z \leqslant y$ and $\langle z\rangle \leqslant \langle x\rangle$. We can show $z \leqslant x$ by showing $z \leqslant \langle x\rangle \cdot y$. This follows from $z \leqslant y$ and $z \leqslant \langle z\rangle \leqslant \langle x\rangle$.

Conversely, from $\langle x\rangle \cdot y \leqslant y$ and $\langle\langle x\rangle \cdot y\rangle \leqslant \langle x\rangle \cdot \langle y\rangle = \langle x\rangle$ (since $x \leqslant y$), we have that $\langle x\rangle \cdot y$ satisfies the hypotheses for z on the right hand side. Thus $\langle x\rangle \cdot y \leqslant x$. To obtain $x \leqslant \langle x\rangle \cdot y$ we use $x \leqslant \langle x\rangle$ and $x \leqslant y$. □

Theorem 6. *For all regions x, y: $\langle x\rangle \cdot \langle y\rangle = \langle x\rangle \implies \langle\langle x\rangle \cdot y\rangle = \langle x\rangle$.*

Proof. Clearly $\langle\langle x\rangle \cdot y\rangle \leqslant \langle x\rangle \cdot \langle y\rangle = \langle x\rangle$, so we just have to show that $\langle x\rangle \leqslant \langle\langle x\rangle \cdot y\rangle$.

Writing y as $(\langle x\rangle^* \cdot y) + (\langle x\rangle \cdot y)$ we get $\langle y\rangle = \langle\langle x\rangle^* \cdot y\rangle + \langle\langle x\rangle \cdot y\rangle$. Now, $\langle\langle x\rangle^* \cdot y\rangle \leqslant \langle\langle x\rangle^*\rangle \cdot \langle y\rangle = \langle x\rangle^* \cdot \langle y\rangle$ using axiom **LCL-3**. Thus $\langle x\rangle$ intersects $\langle\langle x\rangle^* \cdot y\rangle$ in null, since this is its intersection with $\langle x\rangle^* \cdot \langle y\rangle$. Hence we find $\langle x\rangle = \langle x\rangle \cdot \langle y\rangle = \langle x\rangle \cdot \langle\langle x\rangle \cdot y\rangle$ (using the expression for $\langle y\rangle$ above). This establishes that $\langle x\rangle \leqslant \langle\langle x\rangle \cdot y\rangle$. □

Corollary 7. *Let x, y be regions where $x \leqslant_{\mathsf{hist}} y$. Then $\langle x\rangle \cdot y$ is a temporal part of y and is historically equivalent to x. That is, $x \leqslant_{\mathsf{hist}} \langle x\rangle \cdot y$ and $\langle x\rangle \cdot y \leqslant_{\mathsf{hist}} x$.*

5 Temporal Ordering

5.1 Post-History and Pre-History

We use the term *post-history* to denote the region of space-time determined by the period from the last existence of the entity onwards in time. The *pre-history* is the region of space-time extending from the beginning of time to start of the entity's existence.

From these two primitive operators, other important ones can be derived: extended history, extended pre-history and extended post-history. The extended history is the part of space-time during the period from the first existence of the entity to the last. This period takes no account of gaps when the entity may temporarily not exist, so it will contain the historical closure but possibly more in addition. The extended pre-history is the pre-history together with the extended history, and the extended post-history is the post-history together with the extended history.

To form an intuitive picture of the various operators the reader will find it helpful to refer back to figure 1, and to draw further diagrams in the same style.

5.2 The Galois Connection

Given partially ordered sets $(A, \leqslant_A)$ and $(B, \leqslant_B)$, a Galois connection is a pair of functions $f : A \to B$ and $g : B \to A$ such that for all $a \in A$ and all $b \in B$ the statements $a \leqslant_A g(b)$ and $b \leqslant_B f(a)$ are equivalent. Galois connections are closely related to adjunctions and more details can be found, for instance, in [13, p151]. In our case there is just one partially ordered set, the Boolean algebra, A, of spatio-temporal regions partially ordered by the part-of relation. The two functions are the pre-history $\mathsf{pre} : A \to A$, and the post-history $\mathsf{post} : A \to A$. The axiom **GAL** states that these form a Galois connection.

GAL For all x, y, $x \leqslant \mathsf{pre}(y)$ if and only if $y \leqslant \mathsf{post}(x)$.

From the pre- and post-history we can define the extended pre- and extended post-history.

Definition 6 (Extended Pre- and Post-History). *For any region x, the extended pre-history, $\mathsf{xpre}(x)$, and the extended post-history, $\mathsf{xpost}(x)$ are defined by*

$$\begin{aligned} \mathsf{xpre}(x) &= \mathsf{pre}(\mathsf{post}(x)), \\ \mathsf{xpost}(x) &= \mathsf{post}(\mathsf{pre}(x)). \end{aligned}$$

Thus the extended pre-history of x is obtained by first taking the post-history of x (everything after the end of x), and then taking the pre-history of this. This gives everything before the end of x, that is the extended pre-history. Similarly the extended post-history is obtained by first constructing the pre-history, and then taking the post-history of this.

The axiom **GAL** has some straightforward consequences which we make use of later. These are all well-known results about Galois connections and can be proved easily directly from the axiom.

Lemma 8. *The following hold for all regions x:*

1. $x \leqslant \mathsf{post}(\mathsf{pre}(x))$ *and* $x \leqslant \mathsf{pre}(\mathsf{post}(x))$,
2. $\mathsf{post}(\mathsf{pre}(\mathsf{post}(x))) = \mathsf{post}(x)$ *and* $\mathsf{pre}(\mathsf{post}(\mathsf{pre}(x))) = \mathsf{pre}(x)$.

An immediate consequence of the lemma is that the extended pre-history is idempotent, so $\mathsf{xpre}(\mathsf{xpre}(x)) = \mathsf{xpre}(x)$, and similarly for the extended post-history. Another frequently used property which follows from the axiom **GAL**, is that the pre- and post-history are order reversing and that their extended versions are order preserving, or monotone.

Lemma 9. *For any regions, x and y, if $x \leqslant y$ then*

1. $\mathsf{pre}(y) \leqslant \mathsf{pre}(x)$ *and* $\mathsf{post}(y) \leqslant \mathsf{post}(x)$,
2. $\mathsf{pre}(\mathsf{post}(x)) \leqslant \mathsf{pre}(\mathsf{post}(y))$ *and* $\mathsf{post}(\mathsf{pre}(x)) \leqslant \mathsf{post}(\mathsf{pre}(y))$

5.3 Further History Axioms

In addition to the Galois connection axiom, **GAL**, we propose three further axioms.

HST-1 For all x, $\mathsf{xpre}(x) = \mathsf{post}(x)^*$.

HST-2 For all x, $\mathsf{pre}(x) = \mathsf{pre}(\langle x \rangle)$.

HST-3 For all x, y, $x\mathsf{C}y \implies \mathsf{pre}(x) \cdot \mathsf{post}(y) = \mathsf{null}$

These axioms have some immediate consequences, which we state without proof. In Lemma 10 we find the dual version (i.e. interchanging pre-history and post-history) of axiom **HST-1**. Lemma 11 shows that the pre-history and post-history of a region are themselves historically closed. This lemma also gives the dual of axiom **HST-2**.

Lemma 10. *For all x,* $\mathsf{xpost}(x) = \mathsf{pre}(x)^*$.

Lemma 11. *For all x,* $\mathsf{post}(x) = \mathsf{post}(\langle x \rangle) = \langle \mathsf{post}(x) \rangle$, *and* $\mathsf{pre}(x) = \langle \mathsf{pre}(x) \rangle$.

Lemma 12. *For all x,* $\mathsf{pre}(x) \leqslant \mathsf{xpre}(x)$ *and* $\mathsf{post}(x) \leqslant \mathsf{xpost}(x)$.

Lemma 13. *For all x,* $x \cdot \mathsf{pre}(x) = \mathsf{null} = x \cdot \mathsf{post}(x)$.

5.4 Temporal Order

An important feature of our use of the Galois connection axiom is its appearance in the definition of temporal ordering. The idea is that entity x temporally precedes y provided x ends no later than y starts. We denote this situation by $x \lhd y$. Using the operators we have introduced, we reduce temporal precedence to a mereological relationship.

Definition 7 (Temporal Precedence). *For all x and y, $x \lhd y$ if and only if $x \leqslant \mathsf{pre}(y)$.*

The definition is equivalent to putting $x \lhd y$ if and only if $y \leqslant \mathsf{post}(x)$, and the two equivalent forms highlight the role of the Galois connection in formalizing the notion that x is before y if and only if y is after x. A further alternative characterization of the $\lhd$ relation is provided in theorem 14.

Theorem 14. *$x \lhd y$ if and only if* $\mathsf{xpre}(x) \leqslant \mathsf{pre}(y)$.

Theorem 15. *$x \lhd y$ and $y \lhd x$ cannot both hold for non-empty regions x and y.*

Proof. If $x \lhd y$ and $y \lhd x$ we get $\mathsf{xpre}(x) \leqslant \mathsf{pre}(y) \leqslant \mathsf{xpre}(y) \leqslant \mathsf{pre}(x)$, by lemma 12, and so $\mathsf{pre}(x) = \mathsf{xpre}(x)$. As $x \leqslant \mathsf{xpre}(x)$ and $x \cdot \mathsf{pre}(x) = \mathsf{null}$, we get $x = \mathsf{null}$. □

Theorem 16. *For all t, x, y, z,* $(x \lhd y \wedge y \between z \wedge z \lhd t) \implies x \lhd t$.

Proof. From $x \lhd y$, we get $\mathsf{xpre}(x) \leqslant \mathsf{pre}(y)$ by theorem 14. From $y \between z$ we get $\mathsf{pre}(y) \leqslant \mathsf{xpre}(z)$ by axioms **HST-1, HST-2, HST-3**. From $z \lhd t$, we get $\mathsf{xpre}(z) \leqslant \mathsf{pre}(t)$. Hence we get $\mathsf{xpre}(x) \leqslant \mathsf{pre}(t)$, and so $x \lhd t$ by theorem 14. □

Theorem 17. *For all x, y; if $x \lhd y$, then for all z (1) $z \leqslant_{\mathsf{hist}} x \implies z \lhd y$ and (2) $z \leqslant_{\mathsf{hist}} y \implies x \lhd z$.*

Proof. For (1), suppose $\mathsf{xpre}(x) \leqslant \mathsf{pre}(y)$ and that $z \leqslant_{\mathsf{hist}} x$. By monotonicity of xpre and lemma 11, we have $\mathsf{xpre}(z) \leqslant \mathsf{xpre}(x)$, and hence $\mathsf{xpre}(z) \leqslant \mathsf{pre}(y)$. Part (2) is similar. □

The last theorem in this section follows easily from the definition of $\lhd$.

Theorem 18. *For all x, y, z, $(x \lhd y \wedge z \lhd y)$ if and only if $(x + z) \lhd y$.*

6 Comparison with Muller's approach

Our work is related to Muller's axiomatization [7], and although the two approaches are not equivalent, a comparison does highlight some of the advantages in using the algebraic approach. Muller states 29 axioms, of which the first 26 correspond to the approach in the present paper. Of axioms A1 to A26, the first ten are purely mereotopological. In our approach the role of these ten is played by the axioms for a Boolean connection algebra, but the two are not equivalent as Muller is able to refer to interiors and closures of regions, which are not part of the BCA formulation.

Mullers axioms A11 to A26 play the same role as our seven axioms **LCL-1 – LCL-3**, **GAL**, and **HST-1 – HST-3**. Mullers A21 and A22 simply serve to rule out degenerate cases and could be imposed as additional requirements on our system if necessary. A25 is only relevant where the mereotopological theory includes a notion of interior. A13 states that $x \between y$ implies that x is not temporally before y. Our motivation for not requiring this comes from considering x and y being adjacent periods, say the days Monday and Tuesday. We take these to satisfy $x \between y$ while still saying that Monday precedes Tuesday.

The remaining twelve axioms A11–12, A14–20, A23–24, A26 are all consequences of our seven axioms. Our theorem 2 yields A11, A12, A19, and A24. Lemma 4 proves A20. Axiom A26 is proved by corollary 7, using theorem 5 to justify the equivalence of Muller's "temporal slice" concept TSxy with our condition. Axioms A17 and A18 assert the existence of a 'past' and 'future' for a region x, which are guaranteed in our system by $\mathsf{pre}(x)$ and $\mathsf{post}(x)$.

We have reduced twelve axioms to seven, but simply counting axioms does not provide the full story. The seven are all relatively simple in comparison with some of the twelve (A16 and A26 for example), and the fact that they relate to well known structures (historical closure satisfying the Kuratowski axioms, and pre and post forming a Galois connection) means that the established body of knowledge about these structures can be used in the development of the theory. The greater simplicity of the algebraic approach is seen not only in the axioms but in definitions too. Compare, for example, the succinct $\langle x \rangle \cdot y = x$ with the right hand side of D18 [7, p429] (in our notation): $x \leqslant y \wedge \forall z((z \leqslant y \wedge z \leqslant_{\mathsf{hist}} x) \implies z \leqslant x)$.

7 Conclusions and Further Work

This paper has given the first account of spatio-temporal mereotopology in the algebraic framework. We have shown that by adding two extra structures – a topological closure and a Galois connection – on to a Boolean Connection Algebra allows the definition of many important notions. These defined notions include historical part, historical connection, temporal part, and temporal precedence. We have compared our approach with that of Muller in [7] and shown that a significant simplification has been achieved.

Further work will build on the algebraic foundation presented here and there are a number of specific directions we intend to pursue. One is the development of an algebraic account of qualitative change over time, including the granularity concepts introduced in [14]. Another direction is the replacement of the Boolean Connection algebra used here with a formulation which permits discrete space.

Acknowledgements

The first author was supported by EPSRC through the project *Digital Geometry and Topology: An Axiomatic Approach with applications to GIS and Spatial Reasoning*. We are grateful to Pierre Grenon for some detailed comments, and to the anonymous reviewers for their suggestions.

References

[1] J. G. Stell. Boolean connection algebras: A new approach to the region-connection calculus. *Artificial Intelligence*, 122:111–136, 2000.

[2] D. Vakarelov, G. Dimov, I. Düntsch, and B. Bennett. A proximity approach to some region-based theories of space. *Journal of Applied Non-Classical Logics*, 12:527–559, 2002.

[3] I. Düntsch, H. Wang, and S. McCloskey. Relation algebras in qualitative spatial reasoning. *Fundamenta Informaticae*, 39:229–248, 1999.

[4] I. Düntsch, H. Wang, and S. McCloskey. A relation-algebraic approach to the region-connection calculus. *Theoretical Computer Science*, 255:63–83, 2001.

[5] D. A. Randall and A. G. Cohn. Modelling topological and metrical properties in physical processes. In R. J. Brachman, H. J. Levesque, and R. Reiter, editors, *Proceedings of First International Conference on Principles of Knowledge Representation and Reasoning*, pages 357–368. Morgan Kaufmann, 1989.

[6] D. A. Randall, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In *Proceedings of 3rd International Conference on Knowledge Representation and Reasoning, Boston*, 1992.

[7] P. Muller. Topological spatio-temporal reasoning and representation. *Computational Intelligence*, 18:420–450, 2002.

[8] P. Simons. *Parts. A Study in Ontology*. Clarendon Press, Oxford, 1987.

[9] T. Sider. *Four-Dimensionalism - An Ontology of Persistence and Time*. Clarendon Press, Oxford, 2001.

[10] U. Meixner. *Axiomatic Formal Ontology*, volume 264 of *Synthese Library*. Kluwer, 1997.

[11] A. Gallois. *Occasions of Identity*. Clarendon Press, Oxford, 1998.

[12] A. N. Whitehead. *Process and Reality: An Essay in Cosmology*. Cambridge University Press, 1929.

[13] P. Taylor. *Practical Foundations of Mathematics*. Cambridge University Press, 1999.

[14] J. G. Stell. Granularity in change over time. In M. Duckham, M. Goodchild, and M. Worboys, editors, *Foundations of Geographic Information Science*, pages 95–115. Taylor and Francis, 2003.